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Travelling Wave Group-Invariant Solutions and Conservation Laws for θ-Equation

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ABSTRACT

We study a class of nonlinear dispersive models called the θ-equations from the Lie group-theoretic point of view. The Lie point symmetry generators of the class of equations are derived. Using the optimal system of one-dimensional subalgebras constructed from these symmetry generators, we obtain symmetry reductions and travelling wave group-invariant solutions for the underlying equation. Moreover, we construct conservation laws for the class of equations by making use of the partial Lagrangian approach and the multiplier method. The underlying equation is of odd order and thus not variational. To apply the partial variational method a nonlocal transformation $u = v_x$ is used to raise the order of the given class of equations. We show that the existence of nonlocal conservation laws for underlying equation is possible only if $\theta = 1/3$. In the multiplier approach, we obtain conservation laws for the given class of equations and a special case of the equation when $\theta = 1/3$ in which the first-order multipliers are considered.

Keywords: θ-equation; Lie point symmetries; optimal system; group-invariant solutions; partial Lagrangian; multiplier method; conservation laws.

1. Introduction

Nonlinear evolution equations (NLEEs) arise as models of mathematical problems in many branches of engineering, applied mathematics and physics. The investigation of exact solutions of NLEEs plays a central role in the study of these equations. One of the most interesting aspects of study is to determine their integrability since the existence of the exact solutions enables one to have a better understanding of the phenomena modeled by these NLEEs.

Moreover, the understanding of conservation laws also plays an important role in the solution process. Finding the conservation laws of system of partial differential equations (PDEs) is often the first step towards finding the solution. Indeed, the existence of a large number of conservation laws of a system of PDEs is a strong indication of its integrability. Notwithstanding, they have significant uses in the study of the analysis of stability and global behaviour of solutions and in developing numerical methods for PDEs Bluman and Kumei (1989).

The Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0 \tag{1}$$

had initially been introduced by the authors Fokas and Fuchssteiner (1981) as a bi-Hamiltonian system with infinitely many conservation laws. Subsequently, the equation (1) was derived from physical principles as a mathematical model of shallow water waves in Camassa and Holm (1993). There have been various generalized forms of the Camassa-Holm equation (1) which have been studied by the authors Dai (1998), Guo and Tang (2014), Lai et al. (2013), Niu and Zhang (2011), Wu and Yin (2009), and so on.

In this paper we study one of such generalizations of the Camassa-Holm equation (1), a class of nonlinear dispersive models known as the θ-equations of the form

$$u_t - u_{txx} + uu_x - (1-\theta)u_x u_{xx} - \theta uu_{xxx} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2}$$

where $u(t,x)$ denotes the velocity field at time t in the spatial x direction. The equation (2) has been reported in the works (see e.g., Liu (2008), Zhenshu Wen (2012) and the references therein). When $\theta = 1/3$, the equation (2) becomes

$$3u_t - 3u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{3}$$

and by making use of the transformation

$$\widetilde{t} = \frac{t}{3} \quad \widetilde{x} = x, \quad \widetilde{u} = u, \tag{4}$$

the equation (3), becomes the Camassa-Holm equation (1).

Our aim in the present work is to obtain by means of optimal system of one-dimensional subalgebras the exact closed-form solutions and reduction for (2) by applying the classical Lie symmetry method. Symmetry approach has proved to be an efficient tool that enables us to develop exact solutions of a given system of PDEs in a systematic manner. For the theory and application of the Lie symmetry methods (see e.g., Bluman and Kumei (1989), Ibragimov (1996), Olver (1993), Ovsiannikov (1982)). We obtain all possible Lie point symmetries of the equation (2). Then, with the help of these symmetries we construct the optimal system of one-dimensional subalgebras and hence symmetry reductions and the travelling wave group-invariant solutions for (2).

Secondly, we construct local and nonlocal conservation laws for the equation (2) by using the partial Lagrangian approach Kara and Mahomed (2006), Naz et al. (2008) and the multiplier method Anco and Bluman (2002), Naz et al. (2009), Olver (1993), Steudel (1962). The partial Lagrangian approach is easier and the application is similar to that of Noether approach Naz et al. (2008). The equation (2) does not have a usual Lagrangian, so in order to utilize the partial variational method to (2), we consider a fourth-order PDE which is obtained by the application of the nonlocal transformation $u = v_x$ to the underlying class of equations. We show that a number of nonlocal conservation laws for (2) exist only when $\theta = 1/3$. In the multiplier approach to construct conservation laws for (2), and for the special case of the equation (3), the first-order multipliers are considered.

The outline of the paper is as follows. In Section 2, we present the main operators and briefly discuss the two approaches in the construction of the conservation laws for the class of equations (2). The Lie point symmetries are obtained in Section 3. In Section 4, we construct the optimal system of one-dimensional subalgebras of the Lie algebra of (2). The Section 5 contains reductions and exact solutions of (2) derived by using the optimal system of one-dimensional subalgebras obtained in Section 4. In Section 6, we construct conservation laws for (2) and the special form of (2), that is, the equation (3), via the two methods outlined in Section 2. Finally, concluding remarks are made in Section 7.

2. Preliminaries

In this section, we present the notation and some important results which are utilized in the following sequel. For details the reader is referred to Anco and Bluman (2002), Bluman and Kumei (1989), Ibragimov (1996), Ibragimov et al. (1998), Kara and Mahomed (2006), Steudel (1962).

Consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, namely

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m, \tag{1}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote the collections of all first, second, ..., kth-order partial derivatives, i.e., $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$, respectively, with the total derivative operator with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \quad i = 1, \ldots, n, \tag{2}$$

where the summation convention is used whenever appropriate.

A Lie-Bäcklund operator in infinite formal sum is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 i_2 \ldots i_s}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \ldots i_s}^\alpha}, \tag{3}$$

where $\xi^i, \eta^\alpha \in \mathcal{A}$ and $\mathcal{A}$ is the space of differential functions. The additional coefficients $\zeta_{i_1 i_2 \ldots i_s}^\alpha$ are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 \ldots i_s}^\alpha &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \ldots i_s}^\alpha, \quad s > 1, \end{aligned} \tag{4}$$

in which $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$ is the Lie characteristic function.

The n-tuple vector $T = (T^1, T^2, \ldots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \ldots, n$, is a conserved vector of (1) if T^i satisfies

$$D_i T^i|_{(1)} = 0. \tag{5}$$

The equation (5) is called a local conservation law of system (1).

The following results are based on the work in Kara and Mahomed (2006).

If the system of equations (1) can be written as

$$E_\alpha = E_\alpha^0 + E_\alpha^1 = 0, \ \alpha = 1, \ldots, m, \tag{6}$$

and if there exists a function $L = L(x, u, u_{(1)}, \ldots, u_{(l)}) \in \mathcal{A}$, $l \leq k$ and nonzero functions $f_\alpha^\beta \in \mathcal{A}$ such that (6) can be written as $\delta L / \delta u^\alpha = f_\alpha^\beta E_\beta^1$, provided

$E^1_\beta \neq 0$, then L is called as a partial Lagrangian of (6). Here, $\delta/\delta u^\alpha, \alpha = 1, \ldots, m$, is a Euler-Lagrange operator and is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\ldots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \qquad \alpha = 1, \ldots, m. \tag{7}$$

Let us write the operator in (3) in the following simple form given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \tag{8}$$

Then the operator X in (8) is called a partial Noether operator corresponding to a partial Lagrangian $L \in \mathcal{A}$ of the system (6) if it can be determined from

$$X(L) + LD_i(\xi^i) = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i(B^i), \tag{9}$$

for some vector $B = (B^1, B^2, \ldots, B^n),\ B^i \in \mathcal{A}$. Here $W = (W^1, W^2, \ldots, W^m),\ W^\alpha \in \mathcal{A}$ is the characteristic of X.

Theorem 2.1 (Kara and Mahomed (2006)). If the operator X as in (8) is a partial Noether operator of a partial Lagrangian L corresponding to a system of the form (6) then the components T^i of the conserved vector T can be constructed by the formula

$$T^i = B^i - \xi^i L - W^\alpha \frac{\delta L}{\delta u^\alpha_i} - \sum_{s\geq 1} D_{i_1}\ldots D_{i_s}(W^\alpha)\frac{\delta L}{\delta u^\alpha_{i i_1 i_2 \ldots i_s}},\ i = 1, \ldots, n, \tag{10}$$

where the characteristic $W = (W^1, W^2, \ldots, W^m), W^\alpha \in \mathcal{A}$ of X is also the characteristic of the conservation law $D_i T^i = 0$ of (1).

It can be shown that every admitted conservation law arises from multipliers $Q^\alpha(x, u, u_{(1)}, \ldots)$ such that

$$Q^\alpha E_\alpha = D_i T^i, \tag{11}$$

holds identically Olver (1993), Steudel (1962). In the multiplier approach for conservation laws, one takes the variational derivative of (11), that is,

$$\frac{\delta}{\delta u^\beta}(Q^\alpha E_\alpha) = 0, \tag{12}$$

holds for arbitrary functions of $u(x^1, x^2, \ldots, x^n)$, (see also, Anco and Bluman (2002), Olver (1993)). All the multipliers can be derived from the determining equation (12) for which the underlying equation is expressed as a local conservation law.

3. Lie point symmetry generators of (2)

In this section, we derive the Lie point symmetries admitted by the equation (2).

A vector field

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{1}$$

is a generator of point symmetry of the equation (2) if

$$X^{[3]}[u_t - u_{txx} + uu_x - (1-\theta)u_x u_{xx} - \theta u u_{xxx}]|_{(2)} = 0, \tag{2}$$

where the operator $X^{[3]}$ is the third prolongation of the operator X defined by

$$X^{[3]} = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xxx}\frac{\partial}{\partial u_{xxx}} + \zeta_{txx}\frac{\partial}{\partial u_{txx}},$$

the coefficients $\zeta_t, \zeta_x, \zeta_{xxx}$ and ζ_{txx} are given by

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
\zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
\zeta_{xx} &= D_x(\zeta_x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi),\\
\zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi),\\
\zeta_{txx} &= D_t(\zeta_{xx}) - u_{xxt} D_t(\tau) - u_{xxx} D_t(\xi).
\end{aligned}$$

Here D_i is the total derivative operator as given in (2).

The coefficient functions τ, ξ and η are independent of the derivatives of u, thus by equating the coefficients of like derivatives of u in the determining equation (2) yields the following over determined system of linear PDEs:

$$\tau = \tau(t),\ \xi = \xi(t,x),\ \eta_{uu} = 0, \tag{3}$$

$$\xi_{xx} - 2\eta_{xu} = 0,\ 2\xi_x - \eta_{xxu} = 0,\ -3\xi_x + \tau_t + \eta_u + \eta_{xxu} = 0, \tag{4}$$

$$\eta_t + \eta_x u - \eta_{txx} - \theta\,\eta_{xxx} u = 0, \tag{5}$$

$$(\theta-1)\eta_x + 2\xi_{tx} + 3\theta\xi_{xx}u - \eta_{tu} - 3\theta\eta_{xu}u = 0, \tag{6}$$

$$\xi_t - \theta\,\eta + 3\theta\,\xi_x u - \theta\tau_t u - \theta\eta_{xxu}u = 0, \tag{7}$$

$$\eta - \xi_t - \xi_x u + \tau_t u + (\theta-1)\eta_{xx} + \xi_{txx} + \theta\xi_{xxx}u - 2\eta_{txu} + \eta_{xxu}u - 3\theta\eta_{xxu}u = 0. \tag{8}$$

Solving the determining equations (3)-(8) for τ, ξ and η, we obtain the following Lie point symmetry generators admitted by the equation (2)

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = t\,\frac{\partial}{\partial t} - u\,\frac{\partial}{\partial u}.$$

4. Optimal system of subalgebras of the equation (2)

In order to obtain symmetry reductions and to construct classes of group-invariant solutions for the equation (2) in a systematic manner, we derive optimal system of one-dimensional subalgebras for (2). The method used for constructing an optimal system of one-dimensional subalgebras is the one that was given in Olver (1993).

To calculate the adjoint representation, we use the following well-known Lie series.

$$\mathrm{Ad}(\exp(\mathrm{e}X))Y = Y - \mathrm{e}[X,Y] + \frac{1}{2!}\mathrm{e}^2[X,[X,Y]] - \frac{1}{3!}\mathrm{e}^3[X,[X,[X,Y]]] + ...,$$

together with the commutation relations of the three operators which are $[X_1, X_2] = 0$, $[X_1, X_3] = X_1$, $[X_2, X_3] = 0$.

For example,

$$\begin{aligned}\mathrm{Ad}(\exp(\mathrm{e}X_1))X_3 &= X_3 - \mathrm{e}[X_1, X_3] + \frac{1}{2!}\mathrm{e}^2[X_1,[X_1,X_3]] - ... \\ &= X_3 - \mathrm{e}X_1. \end{aligned} \quad (1)$$

Similarly, we can find the other entries of the adjoint table. We thus have the adjoint representation given by the table below.

Table I: The adjoint table for the symmetries

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \mathrm{e}X_1$
X_2	X_1	X_2	X_3
X_3	$\exp(\mathrm{e}X_1)$	X_2	X_3

In the above, the (i, j) entry represents $\mathrm{Ad}(\exp(\mathrm{e}X_i))X_j$. For a nonzero vector

$$X = a_1X_1 + a_2X_2 + a_3X_3,$$

we have to to simplify the coefficients a_i as far as possible through adjoint maps to X. The calculations are easy and we obtain an optimal system of one-dimensional subalgebras spanned by

$$\begin{aligned} &X_1, \\ &X_2, \\ &X_1 \pm X_2, \\ &aX_2 + X_3. \end{aligned} \tag{2}$$

5. Reduction and group-invariant solutions of the equation (2)

In this section we make use of the optimal system of one-dimensional of subalgebras of the equation (2), found in Section 4, to derive reductions and exact group-invariant solutions for the equation (2).

(a) X_2.

The group-invariant solution corresponding to X_2 is $u = h(\gamma)$, where $\gamma = t$ is the group invariant of X_2. The substitution of this solution into the equation (2) and solving the ordinary differential equation (ODE) we obtain the solution for (2) given by $u(t, x) = C$, here C is a constant.

(b) $X_1 + \epsilon X_2$, where $\epsilon = 0, \pm 1$ (Travelling wave solutions).

Here, the group-invariant solution is given by $u(t, x) = h(\gamma)$, where $\gamma = x - \epsilon t$ is the group invariant of $X_1 + \epsilon X_2$. Substitution of this solution into the equation (2) gives rise to the ODE

$$(\theta h - \epsilon)h''' + (1 - \theta)h'h'' + \epsilon h' - hh' = 0, \tag{1}$$

here 'prime' denotes differentiation with respect to γ. Differentiating the equation (1) twice with respect to γ we obtain the following nonlinear first-order ODE

$$(h')^2 = \frac{C_1}{1 - 2\theta} + h^2 + C_2(\theta h - c)^{\frac{2\theta - 1}{\theta}}, \tag{2}$$

where C_1 and C_2 are arbitrary constants of integration. The equation (2) is highly nonlinear, therefore in order to find solutions of (2), we consider the following cases.

Case 1 ($C_1 = C_2 = 0$).

In this case, the equation (2) turns out to be

$$(h')^2 = h^2. \tag{3}$$

Integration of the equation (3) yields the following group-invariant solution of (2) given by

$$u(t,x) = A\ \exp[\pm(x - \epsilon t)], \tag{4}$$

where A is a nonzero constant.

Case 2 ($C_2 = 0$).

Here, the equation (2) becomes

$$(h')^2 = B^2 + h^2, \tag{5}$$

where $B = \pm\sqrt{C_1/(1-2\theta)}$ and $\theta \neq 1/2$. Solving the equation (5) gives rise to the following group-invariant solution of (2)

$$u(t,x) = \pm\sqrt{\frac{C_1}{(1-2\theta)}}\ \sinh[\pm(x - \epsilon t) + \delta], \tag{6}$$

where δ is a constant.

(c) $X_3 + aX_2$, a is a constant.

This symmetry generator gives rise to the group-invariant solution $u(t,x) = 1/t\, h(\gamma)$, where $\gamma = x - a\ \ln t$ is the group invariant of $X_3 + aX_2$. Substitution of this solution into the equation (2) results in the ODE

$$(a - \theta h)h''' - (1-\theta)h'h'' + h'' + (h - a)h' - h = 0, \tag{7}$$

here 'prime' denotes differentiation with respect to γ. The equation (7) has a Lie point symmetry $\partial/\partial\gamma$. The inter-change transformation, $u(\gamma,h) = h,\ \ v(\gamma,h) = \gamma$, will transform $\partial/\partial\gamma$ to $\partial/\partial v$. By choosing the the transformations, $w = h' = (dv/du)^{-1},\ h'' = wdw/du$ and $h''' = w^2(d^2w/du^2) + w(dw/du)^2$, the equation (7) can further be reduced as

$$(a-\theta u)w^2\frac{d^2w}{du^2} + (a-\theta u)w\left(\frac{dw}{du}\right)^2 + [1-(1-\theta)w]w\frac{dw}{du} + (u-a)w - u = 0. \tag{8}$$

If the equation (8) has a solution, $w = f(u, C_1, C_2)$, where C_1 and C_2 are arbitrary constants, then the general solution of (7) is given by

$$\gamma + C_3 = \int^{r=u(\gamma,h)} \frac{dr}{f(r, C_1, C_2)}, \tag{9}$$

where C_3 is an arbitrary constant.

6. Conservation laws

In this section, we construct local and nonlocal conservation laws for the equation (2) and (3), by using the two different approaches as outlined in Section 2.

6.1. *Partial variational approach*

The equation (2) is an evolution equation and of odd order. Thus it does not have a Lagrangian nor a partial Lagrangian. So we make use of the substitution $u = v_x$, so that the equation (2) becomes

$$v_{xt} - v_{txxx} + v_x v_{xx} - (1-\theta)v_{xx}v_{xxx} - \theta v_x v_{xxxx} = 0, \tag{1}$$

which then has a partial Lagrangian given by

$$L = -\frac{1}{2}v_x v_t - \frac{1}{2}v_{tx}v_{xx} - \frac{1}{6}v_x^3 + \frac{\theta}{4}v_x^2 v_{xxx}. \tag{2}$$

So we have

$$\frac{\delta L}{\delta v} = (1-3\theta)v_{xx}v_{xxx}. \tag{3}$$

Let us assume the partial Noether operator is of the form as given in (1). Then by separating the monomials in the partial Noether operator determining equation (9) with the partial Lagrangian (2), we obtain the following determining equations for the coefficients τ, ξ and η:

$$\tau_v = 0,\ \xi_v = 0, \tag{4}$$

$$(1-3\theta)\tau = 0,\ (1-3\theta)\xi = 0,\ (1-3\theta)\eta = 0, \tag{5}$$

$$-\theta\tau_x = 0,\ \eta_v = 0,\ \xi_t = 0, \tag{6}$$

$$\theta(\tau_t - 4\xi_x) = 0 \tag{7}$$

$$\theta\eta_x = 0,\ \xi_x = 0,\ \tau_t = 0, \tag{8}$$

$$B_v^1 = 0,\ B_v^2 = -\frac{\eta_t}{2} \tag{9}$$

$$B_t^1 + B_x^2 = 0. \tag{10}$$

Solving the equations (4)-(10) leads to a nontrivial solution only when $\theta = 1/3$. Then from (3) we deduce that, the partial Lagrangian is indeed a Lagrangian for (1) and the generators are the corresponding Noether symmetries. The

Noether symmetries and the gauge functions for the equation (1) are given by

$$X_1 = \frac{\partial}{\partial t},\ B^1 = 0,\ B^2 = 0 \tag{11}$$

$$X_2 = \frac{\partial}{\partial x},\ B^1 = 0,\ B^2 = 0 \tag{12}$$

$$X_3 = a(t)\,\frac{\partial}{\partial v},\ B^1 = 0,\ B^2 = -\frac{1}{2}a_t v. \tag{13}$$

We list the corresponding conserved vectors which are obtained by invoking (10) as follows:

(i) $X_1 = \partial/\partial t,\ B^1 = 0,\ B^2 = 0.$

$$\begin{aligned}
T^1 &= \frac{1}{6}v_x^3 - \frac{1}{12}v_x^2 v_{xxx} + \frac{1}{2}v_t v_{xxx},\\
T^2 &= -\frac{1}{2}v_t^2 - \frac{1}{2}v_t v_x^2 + \frac{1}{3}v_t v_x v_{xxx} + v_t v_{xxt} + \frac{1}{6}v_t v_{xx}^2 - \frac{1}{2}v_{tt}v_{xx} - \frac{1}{2}v_{xt}^2 - \frac{1}{6}v_x v_{xt} v_{xx} + \frac{1}{12}v_x^2 v_{xx}
\end{aligned}$$

Thus

$$\begin{aligned}
D_t T^1 + D_x T^2 &= -v_t\left(v_{xt} - v_{txxx} + v_x v_{xx} - \frac{2}{3}v_{xx}v_{xxx} - \frac{1}{3}v_x v_{xxxx}\right) + D_t\left(\frac{1}{2}v_t v_{xxx}\right)\\
&\quad + D_x\left(-\frac{1}{2}v_{tt}v_{xx}\right).
\end{aligned}$$

Thus we obtain the following simplified form for the components $\widetilde{T}^1$ and $\widetilde{T}^2$, namely,

$$\begin{aligned}
\widetilde{T}^1 &= \frac{1}{6}v_x^3 - \frac{1}{12}v_x^2 v_{xxx},\\
\widetilde{T}^2 &= -\frac{1}{2}v_t^2 - \frac{1}{2}v_t v_x^2 + \frac{1}{3}v_t v_x v_{xxx} + v_t v_{xxt} + \frac{1}{6}v_t v_{xx}^2 - \frac{1}{2}v_{xt}^2 - \frac{1}{6}v_x v_{xt} v_{xx} + \frac{1}{12}v_x^2 v_{xxt},
\end{aligned}$$

which are now the components of the conserved vector $\widetilde{T}$ for the equation (1) with $\theta = 1/3$, and satisfies the equation $D_i\widetilde{T}^i|_{(1)} = 0$.

(ii) $X_2 = \partial/\partial x,\ B^1 = 0,\ B^2 = 0.$

$$\begin{aligned}
T^1 &= -\frac{1}{2}v_x^2 + \frac{1}{2}v_x v_{xxx} - \frac{1}{2}v_{xx}^2\\
T^2 &= -\frac{1}{3}v_x^3 + \frac{1}{3}v_x^2 v_{xxx} - \frac{1}{2}v_{xt}v_{xx} + v_x v_{xxt}.
\end{aligned}$$

So that

$$\begin{aligned}
D_t T^1 + D_x T^2 &= -v_x\left(v_{xt} - v_{txxx} + v_x v_{xx} - \frac{2}{3}v_{xx}v_{xxx} - \frac{1}{3}v_x v_{xxxx}\right) + D_t\left(-\frac{1}{2}v_{xx}^2\right)\\
&\quad + D_x\left(\frac{1}{2}v_x v_{xxt}\right).
\end{aligned}$$

This gives rise to the new components $\widetilde{T}^1$ and $\widetilde{T}^2$ given by

$$\begin{aligned}\widetilde{T}^1 &= -\frac{1}{2}v_x^2 + \frac{1}{2}v_x v_{xxx},\\ \widetilde{T}^2 &= -\frac{1}{3}v_x^3 + \frac{1}{3}v_x^2 v_{xxx} - \frac{1}{2}v_{xt}v_{xx} + \frac{1}{2}v_x v_{xxt},\end{aligned}$$

One can readily verify that the equation $D_i\widetilde{T}^i|_{(1)} = 0$ is satisfied for $\theta = 1/3$.

(iii) $X_3 = a(t)\,\partial/\partial v,\ B^1 = 0,\ B^2 = -1/2a_t v.$

$$\begin{aligned}T^1 &= \frac{1}{2}av_x - \frac{1}{2}av_{xxx}\\ T^2 &= -\frac{1}{2}a'v + \frac{1}{2}av_t + \frac{1}{2}av_x^2 - \frac{1}{3}av_x v_{xxx} - \frac{1}{2}av_{xxt} - \frac{1}{6}\,av_{xx}^2 + \frac{1}{2}a'v_{xx}.\end{aligned}$$

It can also be readily verified that for the equation (1) with $\theta = 1/3$, $D_iT^i|_{(1)} = 0$.

Remark 1. The conserved vectors (i) and (iii) are nonlocal conserved vectors for the equation (2) when $\theta = 1/3$, that is, the equation (3), as by the transformation $u = v_x$, $v_t = \int u_t\,dx$. Using this transformation, the conserved vector in (ii) becomes $\widetilde{T}^1 = -\frac{1}{2}u^2 + \frac{1}{2}uu_{xx},\ \widetilde{T}^2 = -\frac{1}{3}u^3 + \frac{1}{3}u^2u_{xx} - \frac{1}{2}u_tu_x + \frac{1}{2}uu_{tx}$, which is a conservation laws for the equation (3).

6.2. *Multiplier approach*

6.2.1. Conservation laws for (2) when $\theta \neq 1/3$

We consider the multipliers of the form $Q(t, x, u, u_t, u_x)$ for the equation (2). The determining equation (12) for the multipliers takes the form

$$\frac{\delta}{\delta u}[Q(u_t - u_{txx} + uu_x - (1-\theta)u_xu_{xx} - \theta uu_{xxx})] = 0. \tag{14}$$

Expanding (14) yields

$$\begin{aligned}&Q_u[u_t - u_{txx} + uu_x - (1-\theta)u_xu_{xx} - \theta uu_{xxx}] - D_t[Q_{u_t}(u_t - u_{txx} + uu_x\\ &-(1-\theta)u_xu_{xx} - \theta uu_{xxx})] - D_x[Q_{u_x}(u_t - u_{txx} + uu_x - (1-\theta)u_xu_{xx}\\ &-\theta uu_{xxx})] + Q(u_x - \theta u_{xxx}) - D_t(Q) - D_x[Q(u - (1-\theta)u_{xx})] - D_x^2[Q((1-\theta)u_x)]\\ &+D_x^3(\theta Qu) + D_tD_x^2(Q) = 0.\end{aligned} \tag{15}$$

In (15), we equate to zero the coefficients of derivatives of u starting from the highest order and then solving the system of equations which requires detailed calculations of which we find that the multiplier Q takes the form

$$Q = c_1, \quad \text{where } c_1 \text{ is a constant.} \tag{16}$$

The conserved vector (T^1, T^2) of (2) satisfies the divergence relation given by

$$Q[u_t - u_{txx} + uu_x - (1-\theta)u_x u_{xx} - \theta u u_{xxx}] = D_t T^1 + D_x T^2, \tag{17}$$

for all arbitrary functions $u(t,x)$. From (16) and (17), we have

$$c_1[u_t - u_{txx} + uu_x - (1-\theta)u_x u_{xx} - \theta u u_{xxx}] = D_t[c_1(u - u_{xx})]$$
$$+D_x\left[c_1\left(\frac{1}{2}u^2 + \theta u_x^2 - \frac{1}{2}u_x^2 - \theta u u_{xx}\right)\right]. \tag{18}$$

Thus whenever $u(t,x)$ is solution of (2), we have

$$D_t[c_1(u - u_{xx})] + D_x\left[c_1\left(\frac{1}{2}u^2 + \theta u_x^2 - \frac{1}{2}u_x^2 - \theta u u_{xx}\right)\right] = 0. \tag{19}$$

Hence we derive the following conserved vector for (2) from (19):

$$T^1 = u - u_{xx}, \quad T^2 = \frac{1}{2}u^2 + \theta u_x^2 - \frac{1}{2}u_x^2 - \theta u u_{xx}.$$

6.2.2. Conservation laws for (2) when $\theta = 1/3$

If $\theta = 1/3$, then (2) becomes the equation (3). Again we consider the multipliers of the form $Q(t, x, u, u_t, u_x)$ for the equation (3). The determining equation (12) becomes

$$\frac{\delta}{\delta u}[Q(3u_t - 3u_{txx} + 3uu_x - 2u_x u_{xx} - u u_{xxx})] = 0. \tag{20}$$

Expanding (20), we obtain

$$Q_u(3u_t - 3u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx}) - D_t[Q_{u_t}(3u_t - 3u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx})]$$
$$-D_x[Q_{u_x}(3u_t - 3u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx})] + Q(3u_x - u_{xxx}) - 3D_t(Q)$$
$$-D_x[Q(3u - 2u_{xx})] - 2D_x^2(Qu_x) + D_x^3(Qu) + 3D_t D_x^2(Q) = 0. \tag{21}$$

From (21), by equating to zero the coefficients of derivatives of u and solving the system of equations which involves again some cumbersome calculations we obtain the multiplier Q given by

$$Q = c_1 u + c_2, \tag{22}$$

where c_1 and c_2 are arbitrary constants. The conserved vector (T^1, T^2) of (3) satisfies the divergence relation

$$Q[3u_t - 3u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx}] = D_t T^1 + D_x T^2, \tag{23}$$

for all arbitrary functions $u(t, x)$, that is, from the equations (22) and (23), we have

$$\begin{aligned}
&(c_1 u + c_2)[3u_t - 3u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx}] = D_t\Bigg[c_1\left(\frac{3}{2}u^2 + \frac{1}{2}u_x^2 - uu_{xx}\right) \\
&+c_2(3u - u_{xx})\Bigg] + D_x\Bigg[c_1\left(u^3 - u^2 u_{xx} - 2uu_{tx} + u_t u_x\right) \\
&+c_2\left(\frac{3}{2}u^2 - uu_{xx} - \frac{1}{2}u_x^2 - 2u_{tx}\right)\Bigg].
\end{aligned} \tag{24}$$

Thus if $u(t, x)$ is solution of (3), we have

$$\begin{aligned}
&D_t\left[c_1\left(\frac{3}{2}u^2 + \frac{1}{2}u_x^2 - uu_{xx}\right) + c_2(3u - u_{xx})\right] + D_x\Bigg[c_1\left(u^3 - u^2 u_{xx} - 2uu_{tx} + u_t u_x\right) \\
&+c_2\left(\frac{3}{2}u^2 - uu_{xx} - \frac{1}{2}u_x^2 - 2u_{tx}\right)\Bigg] = 0.
\end{aligned} \tag{25}$$

Hence we derive the following conserved vectors for (3) from (25):

$$T^1 = \frac{3}{2}u^2 + \frac{1}{2}u_x^2 - uu_{xx}, \quad T^2 = u^3 - u^2 u_{xx} - 2uu_{tx} + u_t u_x.$$

and

$$T^1 = 3u - u_{xx}, \quad T^2 = \frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx} - 2u_{tx}.$$

7. Concluding remarks

In this paper, we have constructed exact solutions and conservation laws for a class of nonlinear dispersive equations called the θ-equations in the literature. We used the Lie symmetry method to derive all the Lie point symmetry generators admitted by the equation (2). Then the optimal system of one-dimensional subalgebras of (2) has been used to construct symmetry reductions and closed-form travelling wave group-invariant solutions to the underlying class of equations.

Moreover, we constructed conservation laws for the equation (2) and (3) via the partial variational method and the multiplier approach. The equation

(2) is an evolution one and of odd order, so it is not variational. Then in order to apply the partial variational method to (2), the nonlocal transformation $u = v_x$ was used to raise the order of (2) to the equation (1). In this approach, it is shown that the existence of local and nonlocal conservation laws for (2) is possible only if $\theta = 1/3$. Then from the relation (3) we deduced that, the partial Lagrangian is indeed a Lagrangian for (1) and the generators are the corresponding Noether symmetries. In the multiplier approach to construct conservation laws for (2) and for a special case of the equation (2) when $\theta = 1/3$, that is, the equation (3), the first-order multipliers are considered. For the equation (2), this method gave rise to one multiplier and thus one conserved vector was obtained. For the equation (3), we have two multipliers, so that two conserved vectors have been obtained.

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